JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 2, May 2019
http://dx.doi.org/10.14403/jcms.2019.32.2.191

INJECTIVE HULL OF AN ORE EXTENSION OVER A DIVISION RING

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Abstract. It is shown that a left Ore extension $D[x;\sigma]$ over a division ring D is a left Goldie ring with no zero-divisor and that its left Goldie quotient is an injective hull of $D[x;\sigma]$.

Let D be a division ring and let σ be a nonzero homomorphism from D into itself. Note that σ is a monomorphism since D is a division ring. Denote by $R = D[x;\sigma]$ the left Ore extension over D determined by σ. Refer to [1, Chapter 2] for the left Ore extension. Then R is a free left D-module with basis $\{x^i | i = 0, 1, 2, \ldots\}$ and the multiplication of R satisfies the condition

$$xa = \sigma(a)x \tag{1}$$

for all $a \in D$. Hence every nonzero element $f \in R$ is expressed uniquely by $f = a_n x^n + \ldots + a_0$ for some $a_i \in D$ and $a_n \neq 0$. For such f, we say that f has degree n and denoted by $\deg(f) = n$.

Lemma 1. *For any nonzero elements $f, g \in R$, $fg \neq 0$ and $\deg(fg) = \deg(f) + \deg(g)$. In particular, R has no zero-divisor.*

Proof. Let $f = a_n x^n + \ldots + a_0$ and $g = b_m x^m + \ldots + b_0$, where $a_i, b_j \in D$ and $a_n \neq 0, b_m \neq 0$. Then $fg = a_n \sigma^n(b_m) x^{n+m} +$ (lower terms) by (1). Since σ is a monomorphism and D is a division ring, the leading coefficient $a_n \sigma^n(b_m)$ of fg is nonzero. Hence $fg \neq 0$ and

$$\deg(fg) = n + m = \deg(f) + \deg(g).$$

In particular, R has no zero-divisor. □

Proposition 2. *For any $f, g \in R$ with $f \neq 0$, there exist $q, r \in R$ uniquely such that*

$$g = qf + r,$$

where either $r = 0$ or $\deg(r) < \deg(f)$.

Received January 12, 2019; Accepted March 31, 2019.
2010 Mathematics Subject Classification: [2010]16D50, 16S36.
Key words and phrases: Ore extension, injective hull.

Proof. Repeat the proof of the division algorithm in a polynomial ring over a field. □

COROLLARY 3. *Every left ideal of R is principal.*

Proof. Let N be a left ideal of R. If $N = \{0\}$, then $N = R0$. Suppose that $N \neq \{0\}$ and let f be a nonzero element of N which is of the minimal degree among such elements. For any $g \in N$, there exist $q, r \in R$ such that $g = qf + r$, where either $r = 0$ or $\deg(r) < \deg(f)$ by Proposition 2. If $r \neq 0$ then $r = g - qf \in N$ and $\deg(r) < \deg(f)$, which is a contraction to the minimality of $\deg(f)$. Hence $r = 0$ and $g = qf$. It follows that $N = Rf$, which is principal. □

COROLLARY 4. *The ring R is left noetherian.*

Proof. Every left ideal of R is finitely generated by Corollary 3. Thus R is a left noetherian ring. □

Refer to [1, Chapter 6] for a left Goldie ring and a left Goldie quotient.

COROLLARY 5. *The ring R is a left Goldie ring and thus there exists the left Goldie quotient*

$$Q = \{f^{-1}g | f, g \in R, f \neq 0\}.$$

Proof. It follows by Lemma 1, Corollary 4 and Goldie's theorem [1, Theorem 6.15]. □

Refer to [1, Chapter 5] for the concept of injective hull.

THEOREM 6. *The left Goldie quotient Q is an injective hull of R.*

Proof. Let N be a left ideal of R and let φ be a homomorphism of left R-modules from N into Q. Then $N = Rf$ for some $f \in R$ by Corollary 3. If $\varphi(f) = 0$ then $\varphi = 0$ and thus there exists a homomorphism ψ from R into Q, say $\psi = 0$, such that $\psi|_N = \varphi$. Suppose that $\varphi(f) \neq 0$. Define a map ψ from R into Q by

$$\psi : R \to Q, \quad g \mapsto gf^{-1}\varphi(f)$$

for all $g \in R$. Then ψ is a homomorphism of left R-modules and $\psi(f) = ff^{-1}\varphi(f) = \varphi(f)$ and thus $\psi|_N = \varphi$. Hence Q is an injective left R-module by Baer's Criterion [1, Proposition 5.1].

Every nonzero element $y \in Q$ is of the form $y = f^{-1}g$ for some nonzero elements $f, g \in R$. Hence $0 \neq fy = g \in R$ and thus Q is an essential extension of R. It follows that Q is an injective hull of R. □

Acknowledgments This work was supported by National Research Foundation of Korea, NRF-2017R1A2B4008388, and a research fund of Chungnam National University.

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